CHARACTERISTIC CONSTANTS OF TURBULENCE

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The mixing lengths of turbulent velocity and temperature pulsations are calculated theoretically for a steady flow around an infinitely large plate, also the thickness of the viscous sublayer. The values of these parameters based on analytical formulas are found to be in close agreement with empirical values.

The main characteristics of turbulent flow, including the values of its constants in both the velocity and the temperature distribution equations (namely β_W and β_T in the equations for the mixing lengths of turbulent velocity and temperature pulsations, and the parameter $a = w^*\delta_S/\nu$ describing the thickness δ_S of the viscous sublayer), are related to the diffusion mechanism of propagating turbulent pulsations and, therefore, should be defined on the basis of this mechanism. Usually β_W, a, and β_T are considered to be empirical constants. In this article we will try to calculate them theoretically.

Turbulent pulsations may be treated as flow perturbations traveling along a turbulent stream of fluid. Flow perturbations occur in any stream; the magnitude of a perturbation is characterized by the resulting change in the velocity at a given point in the fluid. In the well-known problem concerning the diffusion of a single straight vortex through a laminar stream, the quantity which characterizes a perturbation is curl w; the analogous parameter in the case of one- or two-dimensional perturbations traveling through a turbulent stream around a flat plate is the derivative of the mean velocity $\partial w_x/\partial z$, which will be denoted here by ω.

The mean equation of a turbulent fluid flowing around a plate in the half-plane xy can be put in the well-known form:

$$\frac{\partial w_x}{\partial \tau} + w_x \frac{\partial w_x}{\partial x} + w_z \frac{\partial w_x}{\partial z} = \nu \left(\frac{\partial^2 w_x}{\partial x^2} + \frac{\partial^2 w_x}{\partial z^2} \right) - \frac{\partial \overline{w_{tx}^2}}{\partial x} - \frac{\partial \overline{w_{tx} w_{tz}}}{\partial z},$$

where w_{tx} and w_{tz} are the longitudinal and the transverse component of the pulsation velocity.

Let us differentiate this equation with respect to z: as a result, we obtain the following equation

$$\frac{\partial \omega}{\partial \tau} + w_x \frac{\partial \omega}{\partial x} + w_z \frac{\partial \omega}{\partial z} + \frac{\partial w_x}{\partial z} \left(\frac{\partial w_x}{\partial x} + \frac{\partial w_z}{\partial z} \right)$$

$$= \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial z^2} \right) + \frac{\partial}{\partial z} \left(- \frac{\partial \overline{w_{tx}^2}}{\partial x} - \frac{\partial \overline{w_{tx} w_{tz}}}{\partial z} \right),$$

where the fourth term on the left-hand side is equal to zero, according to the mean equation of continuity.

If the plate is infinitely large and the flow around it is steady, then the longitudinal component of the mean velocity w_x is a function of only distance z from the plate, i.e., $w_x = w_x(z)$ and the transverse component w_z is equal to zero. Consequently, we have for a steady flow of a turbulent fluid around an infinitely large plate the following equation in ω:

$$\frac{\partial \omega}{\partial \tau} = \frac{\partial}{\partial z} \left[\nu \frac{\partial \omega}{\partial z} + \frac{\partial}{\partial z} (\nu_t \omega) \right], \tag{1}$$

where $\nu_t\omega = \nu_t(\partial w_x/\partial z)$ denotes the quantity $\overline{w_{tx} w_{tz}}$ and ν_t is the turbulent viscosity.

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The turbulent viscosity ν_t varies with the distance from the plate, i.e., is a function of z. In a very rough approximation, ν_t may be considered constant and much larger than ν. The latter assumption is valid when $z \geq \delta_S$; as to the first part of the assumption, it is valid for rather large values of z, i.e., when turbulent pulsations far from the plate are considered. This approximation may serve only as the first step in the analysis, later on it will have to be replaced by a more precise one.

Suppose that a momentary flow perturbation, i.e., a two-dimensional pulsation has been generated at the plate surface (or near it). This pulsation will travel according to the assumption of a constant ν_t so that the equation for ω becomes

$$\frac{\partial \omega}{\partial \tau} = \nu_t \frac{\partial^2 \omega}{\partial z^2} . \tag{2}$$

This equation is identical with the equation of diffusion. Therefore, flow perturbation in the fluid and thus also turbulent velocity perturbation travel by diffusion.

The solution to the equation of diffusion with a constant diffusivity (which happens to be ν_t), for the case of a uniform surface density of diffusing substance (or of momentum in our study) at the plane $z = 0$ (more precisely, the volume density in the adjacent to it layer of thickness Δ) at the initial instant of time, is

$$\omega = \frac{\text{const}}{\sqrt{\tau}} \exp\left(-\frac{z^2}{4\nu_t \tau}\right) . \tag{3}$$

The quantity ω is subject to the constraints: 1) $\omega = \infty$ at $z = 0$ and $\tau = 0$; 2) $\omega = 0$ at $z = \infty$ or $\tau = \infty$.

Expression (3) describes the distribution of ω in space and in time after a momentary pulsation has appeared at $z = 0$ and $\tau = 0$; the expression is valid for large values of z.

We now introduce the mean distance l defined by the relation

$$l = \int_0^\infty \omega z dz \Bigg/ \int_0^\infty \omega dz \ . \tag{4}$$

This expression is analogous to the one used in the kinetic theory of gases for defining the mean free path length of molecules.

Since $\omega = \partial w_x / \partial x$, the integral $\int_0^\infty \omega dz$ represents (accurately, except for the constant factor equal to the fluid density) a momentum associated with the velocity pulsation. Accordingly, l is the effective distance through which a pulsating momentum travels through the fluid. In other words, l represents the effective distance through which a pulsation is transmitted on the average: it is the mixing length of velocity pulsations.

Inserting the earlier value of ω into the expression for l, we have

$$l = \sqrt{\frac{4}{\pi} \nu_t \tau} \ . \tag{5}$$

The quantity ω, when treated as a function of τ at a given z, passes through a maximum which is determined by the condition $(\partial\omega/\partial\tau)_z = 0$ and hence

$$2\nu_t \tau = z^2.$$

This equation defines the position of maximum ω; according to it, the maximum perturbation at the time τ occurs at the distance $z = \sqrt{2\nu_t \tau}$ from the plate (where $z \gg \delta_S$).

With this z in expression (5), the following will be the value of l at the time τ or, which is equivalent, at the distance z from the plate:

$$l = \sqrt{\frac{2}{\pi}} z = 0.8\, z.$$

The magnitude of the numerical coefficient in this equation may not be considered accurate, since the original assumption of a constant ν_t has been based on a rough approximation; besides, solution (3) does not imply that $\int_0^\infty \omega dz = 0$ at $\tau = \infty$. In order to refine the result, we approximate $\nu_t = \alpha z$ on the basis of the established proportionality between l and z, where α is a constant.

In this case the equation for ω will become

$$\frac{\partial \omega}{\partial \tau} = \alpha \frac{\partial^2}{\partial z^2} (z\omega). \tag{6}$$

The solution to this equation for $z \gg \delta_s$ is

$$\omega = \frac{\text{const}}{\tau^2} \exp\left(-\frac{z}{\alpha\tau}\right), \tag{7}$$

as can be easily verified by appropriate substitutions. Inserting the value of ω into the expression for l, we obtain

$$l = \alpha\tau. \tag{8}$$

On the other hand, from the condition $(\partial\omega/\partial\tau)_z = 0$ we have

$$2\alpha\tau = z. \tag{9}$$

With this $\alpha\tau = z/2$, expression (8) yields

$$l = 0.5\,z. \tag{10}$$

It also follows from (9) that the velocity at which the maximum ω moves is $\partial z/\partial\tau = 2\alpha$, i.e., has a constant value; this has to do with the fact that velocity pulsations bring about changes in the velocity field of a moving fluid.

Owing to the constant velocity of the maximum ω and to the finite value of l, which defines the effective path length of a momentum associated with pulsations, it is feasible to treat a turbulent pulsation as an isolated mass or particle of fluid (in the macroscopic sense) moving at the velocity $w^* = 2\alpha$ in a random direction and passing within its life span (i.e., from the instant it appears to the instant it vanishes) through the distance l. This is the basis for the analogy between a turbulent flow and the motion of gas molecules.

Thus, according to Eqs. (10) and (9), the mixing length of a turbulent pulsation is almost one half the distance from the solid wall, while the mixing velocity of a turbulent pulsation is constant.

Experiments yield for $l/z = \beta_w$ approximately 0.4 [1], which agrees closely enough with formula (10). The numerical agreement could possibly be improved still by the use of a higher degree approximation (which, first of all, should involve a replacement of the plane pulsation by a spherical one).

It is to be noted that formula (10), which has been derived analytically, leads to a linear relation between l and z, as once suggested by Prandtl. The proportionality factor β_w between l and z in the well-known Prandtl formula is of fundamental significance in the theory of turbulence, but remains indeterminate and has to be calculated from test data. On the other hand, Eq. (10) yields the acceptable value $\beta_w = 0.5$, which has been established theoretically by an analysis of pulsations diffusing through a turbulent stream of fluid and which represents the kinematic characteristics of turbulent pulsations. Equation (10) offers an interpretation of other peculiarities of a turbulent flow as well, including the proportionality between ν_t and the distance z from the wall. This proportionality is based, first of all, on the proportionality between l and z and, secondly, on the constancy of the pulsation velocity at various points along the stream; by the way, the relation $\nu_t = \alpha z$ can also be arrived at by expanding ν_t into a power series in z and disregarding all terms except the first one.

The logarithmic distribution of mean velocity is explained analogously. Inasmuch as the analysis of a single turbulent velocity pulsation shows that the mixing velocity of pulsations is constant and equal to w^*, while pulsations are continuously generated in the stream, there is a pulsation velocity of the same magnitude at every point in the stream at any instant of time (when the turbulence is anisotropic, the value of the pulsation parameter depends on the direction). Since by definition

$$\nu_t \frac{\partial w_x}{\partial z} = -\overline{w_{tx} w_{tz}},$$

and according to the preceding discussion, $\nu_t = \alpha z$ ($\alpha = w_{tz}/2 = w^*/2$), hence, letting $w_{tx} = -w_{tz}$, we have

$$\frac{w^* z}{2} \cdot \frac{\partial w_x}{\partial z} = w^{*2}.$$

From this follows

$$\frac{\partial w_x}{\partial z} = \frac{2w^*}{z}, \tag{11}$$

i.e., the velocity of the fluid varies logarithmically with the distance z from the plate.

It is to be noted that the expression here already contains the numerical value of the constant β_W (we recall that β_W is the numerical factor in the denominator of the right-hand side of the expression for $\partial w_x/\partial z$), which happens to be equal to 1/2. Thus, in order to determine the numerical value of β_W, it is not absolutely necessary to start out from earlier derived integral formula for l.

If we use the earlier found value $l = z/2$, then the expression for $\partial w_x/\partial z$ may be written as

$$\frac{\partial w_x}{\partial z} = \frac{w^*}{l},$$

from which follows

$$w_{tz} = \frac{\partial w_x}{\partial z} l, \tag{12}$$

by virtue of $w^* = w_{tz}$. Finally, replacing α by $w_{tz}/2$ and z by $2l$ in $\nu_t = \alpha z$, we have

$$\nu_t = w_{tz} l. \tag{13}$$

Expressions (12) and (13) for w_{tz} and ν_t agree with the fundamental semiempirical relations in the theory of turbulence.

We will now calculate the constant $a = w^* \delta_s/\nu$. According to modern concepts, the predominance of molecular viscosity causes the velocity distribution in a viscous sublayer to be the same as in a laminar flow; on the other hand, in a viscous sublayer there occur turbulent velocity pulsations which have penetrated into it from the main stream.

Transverse turbulent pulsations which have penetrated into the viscous sublayer from the main stream should not differ kinematically from perturbations traveling through the viscous sublayer, because the viscous sublayer would otherwise not be stable. In other words, the kinematic characteristics of transverse turbulent pulsations in the viscous sublayer must be the same as those of viscous flow perturbations traveling here: their frequencies at a given point in the sublayer must be the same and the characteristic time, i.e., the time necessary to move across the stream by a distance equal to the sublayer thickness must also be the same.

Owing to the predominance of molecular viscosity, the characteristic time for a viscous sublayer is the same as for a laminar flow.

In the case of a laminar stream of fluid around an infinitely large plate the front edge of which lies on the oy-axis, the equation of a perturbation wave is

$$\frac{\partial \omega}{\partial \tau} + w_x \frac{\partial \omega}{\partial x} + w_z \frac{\partial \omega}{\partial z} = \nu \frac{\partial^2 \omega}{\partial z^2}.$$

If the plate is treated as a continuous source of finitely large perturbations ($\partial\omega/\partial\tau = 0$ for a constantly active source) and if the velocity distribution is taken according to Blasius:

$$w_x = \frac{1.33}{4} w_0 z \sqrt{\frac{w_0}{\nu x}} + \ldots; \quad w_z = \frac{1.33}{16} \cdot \frac{w_0 z^2}{x} \sqrt{\frac{w_0}{\nu x}} + , \ldots,$$

then the solution to the equation for ω with the boundary conditions $\omega = \omega_{wall}(x)$ at $z = 0$ and $\omega = 0$ at $z \geq \delta$ will be

$$\omega = \omega_{wall}\left(1 - \frac{1}{1.5} \int_0^{\xi = \frac{z}{2}\sqrt{\frac{w_0}{\nu x}}} \exp(-0.22\,\xi^2)\, d\xi\right) \tag{14}$$

from where

$$\delta = 3\sqrt{\frac{\nu x}{w_0}}. \tag{15}$$

It is interesting to note that this formula could have been derived directly from the expression for w_x, if it were taken into consideration that velocity w_x must be equal to w_0 at $z = 0$.

Since the ratio x/w_0 represents the sought time τ^* (during this time a perturbation appearing at the edge of the plate is carried by the stream horizontally, i.e., along the plate through the distance x and penetrates normally into the plate, i.e., across the stream through the distance $z = \delta$), hence the characteristic time of flow perturbations in a viscous sublayer is

$$\tau^* = \frac{z^2}{9\nu}. \tag{16}$$

A transverse turbulent pulsation with the velocity w^* would also require the time δ_s/w^* to move through the distance δ, if it moved as in the main stream, while a viscous perturbation requires the time $\delta_s^2/9\nu$. Equating both times, we have

$$\frac{\delta_s}{w^*} = \frac{\delta_s^2}{9\nu}, \tag{17}$$

from which $a = 9$. According to tests, $a = 11.5$ [2].

For $z = \delta_s$, Eq. (2) of viscous perturbations (ν_t replaced by ν) and Eq. (6) of turbulent pulsations must both yield the same value for ω; equating the exponents in (3) and (7), we obtain $\delta_s^2/4\nu = 2\delta_s/w^*$. This indicates that the frequencies of perturbations and pulsations in this analysis are the same, which also yields almost the same value for a.

In conclusion, we consider the mixing length of turbulent temperature pulsations. In a plane-parallel turbulent stream at the same temperature T_0 everywhere (this corresponds to the idealized case of zero thermal flux) and flowing around an infinitely large plate let there appear at the plate surface a momentary two-dimensional temperature pulsation which then travels across the stream.

In the general equation of heat transmission in an incompressible fluid

$$\frac{\partial T_a}{\partial \tau} + w_{ax}\frac{\partial T_a}{\partial x} + w_{az}\frac{\partial T_a}{\partial z} = \frac{\nu}{2c_p}\left(\frac{\partial w_{ai}}{\partial x_j} + \frac{\partial w_{aj}}{\partial x_i}\right)^2 + \operatorname{div} \varkappa \operatorname{grad} T_a$$

we replace the actual temperature T_a by the sum of the mean and the pulsating temperature $T + T_t$ and, with the aid of the continuity equation, we derive the following mean equation for the temperature distribution in a turbulently flowing fluid (we disregard here the terms containing ν and $\varkappa$):

$$\frac{\partial \vartheta}{\partial \tau} = \frac{\partial}{\partial z}\overline{(w_{tz}T_t)}, \tag{18}$$

where $\vartheta = T - T_0$.

In a turbulent stream the heat travels via the ground and via the turbulent velocity pulsations which entrap the entire stream. Since a fast traveling temperature perturbation does not produce a change in the velocity field, or at least does not affect it noticeably, hence the turbulent thermal diffusivity $\varkappa_t = \overline{(w_{tz}T_t)} / (\partial T/\partial z)$ may be assumed equal to βz (β = const), as in the case of turbulent viscosity ν_t. Then the equation for ϑ becomes

$$\frac{\partial \vartheta}{\partial \tau} = \frac{\partial}{\partial z}\left(\beta z \frac{\partial \vartheta}{\partial z}\right).$$

The solution to this equation under conditions analogous to (6) (with $\int_0^\infty \vartheta dz \neq 0$ at $\tau = \infty$) is

$$\vartheta = \frac{\text{const}}{\tau}\exp(-z/\beta\tau). \tag{19}$$

With the aid of this expression, we find from the integral expression for l that

$$l_T = \int_0^\infty \vartheta z dz \Big/ \int_0^\infty \vartheta dz = \beta\tau.$$

On the other hand, from the condition of maximum ϑ, i.e., from the equation $(\partial\vartheta/\partial\tau)_z = 0$ follows

$$-\frac{1}{\tau^2}+\frac{z}{\beta\tau^3}=0$$

or $\beta\tau = z$. Inserting this value of $\beta\tau$ into the expression for l_T will finally yield

$$l_T = z. \tag{20}$$

A comparison between expressions (10) and (20) for l and l_T, respectively, shows that $l_T = 2l$, i.e., that

$$\frac{\beta_T}{\beta_w} = 2. \tag{21}$$

Consequently, the mixing length of turbulent temperature pulsations is greater than that of turbulent velocity pulsations.

This value of the ratio β_T/β_W agrees well enough with the empirical values 1.45-2.0 [3].

It ought to be noted that the difference between the mixing length of velocity and temperature pulsations is not surprising. It is evident already from the kinetic theory of gases that the free path length may be different in terms of internal friction and in terms of thermal conductivity. In a laminar stream the coefficient of downstream heat transfer is 2.5 times greater than the coefficient of momentum transfer to the stream [4]. It is also understandable why the turbulent thermal conductivity exceeds the turbulent viscosity, which, in the final analysis, is attributable to the fact that the differential equations of turbulent velocity and of turbulent temperature pulsation are not identical.

LITERATURE CITED

1. Yu. V. Lapin, Turbulent Boundary Layer in Supersonic Gas Streams [in Russian], Nauka, Moscow (1970).
2. L. G. Loitsyanskii, Mechanics of Liquids and Gases [in Russian], Nauka, Moscow (1970).
3. L. D. Landau and E. M. Lifshits, Mechanics of Continuous Media [in Russian], GITTL, Moscow (1954).
4. A. Sommerfeld, Thermodynamics and Statistical Physics [Russian translation], IL, Moscow (1955).